# Phenomenon of stochastic resonance caused by multiplicative asymmetric dichotomous noise

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The exact expression of the first moment and the signal-to-noise ratio (SNR) have been calculated for a linear system subject to an external periodic field as well as a multiplicative asymmetric dichotomous noise, by using the Shapiro-Loginov formula. It has been found that the amplitude of the output signal, and the SNR, respectively, exhibit two kinds of the phenomena of stochastic resonance: one is as the functions of the parameters of the asymmetric dichotomous noise, such as the noise strength $D$, and the parameter $k$ describing the asymmetric degree of the dichotomous noise; the other is as the function of the parameter of the input signal, such as the input signal frequency $\omega$.

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## I. INTRODUCTION

Noise-induced dynamics has become a center of the subject that has been investigated for a long time in nonlinear statistical systems. One of the phenomena caused by the appearance of the noise is the phenomenon of stochastic resonance (SR) that has attracted considerable interest due to its many applications in biology, physics, chemistry, and other scientific fields [1]. Recently, not only the number of publications on SR phenomenon is growing steadily, but also much extension of the conception of SR has appeared, such as doubly SR [2], stochastic multiresonance [3], coherence resonance (generated in the system without an external force) [4], quantum SR [5], control of SR [6], autonomous SR [7], aperiodic SR [8], etc.

Generally, one believes that the response of the system undergoes resonancelike behavior, and the typical manifestation of SR is the existence of a maximum of the output signal or of the signal-to-noise ratio (SNR) as a function of the noise intensity. But now, besides the general definition, the features of the nonmonotonic dependence of the output signal or SNR on the correlation time of the noise [9], and the frequency of the input signal have also been used to testify SR phenomenon [10].

Three ingredients (i.e., nonlinearity, a periodic signal, and a random force) were thought of as the necessary condition for the appearance of SR. But, it has been later found that SR may appear under the condition in the absence of the periodic signal (such as coherence resonance). Subsequently, the study of the linear system has shown the appearance of SR. It had turned out that SR took place for color multiplicative noise with a short correlation time [11] in the linear system. It was shown that SR also occurred in the linear system for Gaussian colored noise [12], Poissonian noise [13], composite noise [14] (which is intermediate between O-U noise (Gaussian colored noise) and signal-modulated additive colored noise [15]), and correlated noise [9].

In this paper, we will investigate the phenomenon of stochastic resonance induced by the multiplicative asymmetric dichotomous noise in a linear system, from the aspect of the amplitude of the output signal and the signal-to-noise ratio.

## II. OUTPUT SIGNAL

We consider a generic linear stochastic system subject to a sinusoidal signal described by the following stochastic differential equation:

$$\dot{x} = 1 - bx + \xi(t)x + a\sin(\omega t), \quad (1)$$

where $\xi(t)$ represents the asymmetric dichotomous noise, which takes two asymmetric values $-E$, and $kE$, with $E>0$ and $k>0$. $k$ represents the asymmetric degree of the noise. When $k=1$, the noise becomes a symmetric noise. The transition rate from $-E$ to $kE$ is $\gamma$, and the reverse transition rate is $\gamma'$. Without loss of generality, we assume that

$$\langle \xi(t)\rangle = \frac{kE\gamma - E\gamma'}{\gamma+\gamma'} = 0. \quad (2)$$

Thus, we can obtain $k\gamma=\gamma'$. The correlation function of the asymmetric dichotomous noise $\xi(t)$ is

$$\langle \xi(t)\xi(t')\rangle = D\lambda \exp[-\lambda|t-t'|]. \quad (3)$$

Here, $\lambda=(\gamma+\gamma')$ is the reverse of the correlation time $\tau$ of the asymmetric dichotomous noise, and the definition of the strength of the asymmetric dichotomous noise is

$$D = \frac{1}{2}\int_{-\infty}^{+\infty} \langle \xi(\tau)\xi(0)\rangle d\tau = \frac{kE^2}{\lambda}. \quad (4)$$

From Eq. (4), we can find the noise strength $D$ is not independent, but is connected with the asymmetric degree (i.e., $k$), the correlation time $\tau=1/\lambda$, and $E$ of the noise. In Appendix A, we give an example of the application of Eq. (1) with noise $\xi(t)$ whose statistical properties satisfy Eqs. (2)–(4).

In order to make the calculation of the following process be convenient, we first transform the asymmetric dichotomous noise to a one with values $-c$ and $c$. Thus, we assume

$$\xi(t) = \eta(t) + g, \quad (5)$$

where $\eta(t)$ is a dichotomous noise, which takes two values $-c$ and $c$ $(c>0)$. $g$ is a constant. The transition rate from $-c$ to $c$ is $\gamma$, and the reverse transition rate is $\gamma'$. Taking the ensemble average of Eq. (5) and considering Eq. (2), we can obtain

 

$$\langle \eta(t) \rangle = -g. \tag{6}$$

Similarly, we can also derive the correlation function of $\eta(t)$

$$\langle \eta(t)\eta(t') \rangle = g^2 + D\lambda \exp[-\lambda|t-t'|]. \tag{7}$$

Using the relation between the asymmetric dichotomous noise $\xi(t)$ and the dichotomous noise $\eta(t)$, we can get

$$c = \frac{(k+1)E}{2} \quad \text{and} \quad g = \frac{(k-1)E}{2}. \tag{8}$$

In order to calculate the first moment, we average Eq. (1) about $x$ and substitute Eq. (5) into the process of calculation, which gives rise to

$$\frac{d\langle x(t)\rangle}{dt} = 1 - (b-g)\langle x(t)\rangle + \langle \eta(t)x(t)\rangle + a\sin(\omega t). \tag{9}$$

We can find that there is one new correlation factor $\langle \eta(t)x(t)\rangle$ that appeared in Eq. (9). To solve the equation, we will make use of the well-known "formulas of differentiation" [16] that proposed by Shapiro and Loginov, and had been extensively used in Ref. [17]. For the factor $\langle \eta(t)x(t)\rangle$, we have [18]

$$\begin{aligned}\frac{d\langle \eta(t)x(t)\rangle}{dt} &= \left\langle \eta(t)\frac{dx(t)}{dt}\right\rangle - \lambda\langle \eta(t)x(t)\rangle + \lambda(-g)\langle x(t)\rangle \\ &= -[b+\lambda-g]\langle \eta(t)x(t)\rangle + (c^2-\lambda g)\langle x(t)\rangle \\ &\quad - g(1+a\sin(\omega t)],\end{aligned} \tag{10}$$

in which we have used $\eta^2 = c^2$ because of $\eta = \pm c$. We can find that Eqs. (9) and (10) form the closed equations for $\langle x(t)\rangle$ and $\langle \eta(t)x(t)\rangle$. Substituting Eq. (9) into Eq. (10), we can obtain the two-order differential equation about $\langle x(t)\rangle$

$$\begin{aligned}&\frac{d^2\langle x(t)\rangle}{dt^2} + (2(b-g)+\lambda)\frac{d\langle x(t)\rangle}{dt} \\ &\quad + [(b+\lambda-g)(b-g) - c^2 + \lambda g]\langle x(t)\rangle \\ &= a\omega\cos(\omega t) + (b+\lambda-2g)[1+a\sin(\omega t)].\end{aligned} \tag{11}$$

According to the character of the solution of the linear two-order differential equation, the solution of Eq. (11) can be decomposed into two parts. Thus, we can write the solution of Eq. (11) in the following form:

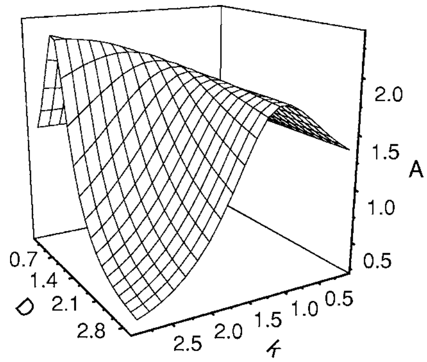


FIG. 1. The amplitude $A$ of the output signal as the function of the noise strength $D$ and the parameter $k$ with $b=1$, $E=1$, $a=1.5$, and $\omega=0.3$.

$$\langle x(t)\rangle = \langle x(t)\rangle_{\text{exter}} + \langle x(t)\rangle_{\text{inter}}, \tag{12}$$

where $\langle x(t)\rangle_{\text{exter}}$ represents the output signal, which is induced by the input signal $a\sin(\omega t)$; and $\langle x(t)\rangle_{\text{inter}}$ is determined by the internal dynamical characteristic of the system. Passing through the detailed calculation, we can obtain the following form:

$$\langle x(t)\rangle_{\text{exter}} = A\sin(\omega t + \varphi),$$

$$\langle x(t)\rangle_{\text{inter}} = \frac{f_4}{f_1+\omega^2}, \tag{13}$$

where

$$A = a\left(\frac{f_3^2+f_4^2}{f_1^2+f_2^2}\right)^{1/2}, \quad \varphi = \arctan\left(\frac{f_1f_3 - f_2f_4}{f_2f_3+f_1f_4}\right), \tag{14}$$

and

$$f_1 = (b+\lambda-g)(b-g) - c^2 + \lambda g - \omega^2, \quad f_3 = \omega,$$

$$f_2 = \omega(2(b-g)+\lambda), \quad f_4 = b+\lambda-2g. \tag{15}$$

From Eq. (14), we can plot the figures for the amplitude $A$ of the output signal vs the parameters of the noise and the input signal. Study shows that the amplitude of the output signal exhibits two kinds of SR phenomena. One is as the function of the parameters of the asymmetric dichotomous noise, such as the noise strength $D$, and the parameter $k$ describing the asymmetric degree of the dichotomous noise [see Figs. 1, 2(a), and 2(b)]. The other is as the function of

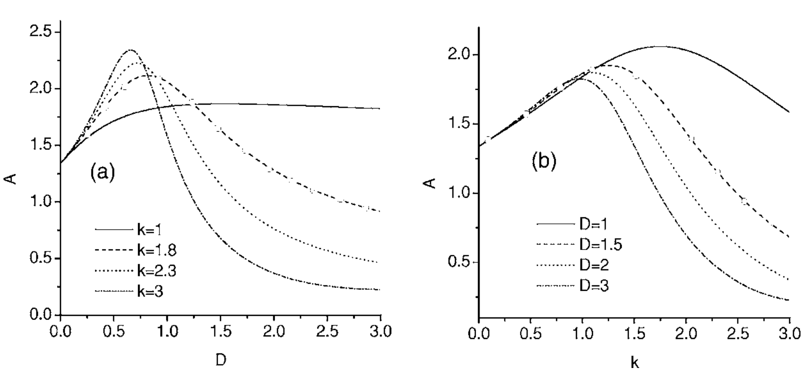


FIG. 2. (a) The amplitude $A$ of the output signal as the function of the noise strength $D$ for different parameters $k=1$, 1.8, 2.3, and 3 with $b=1$, $E=1$, $a=1.5$, and $\omega=0.3$; (b) The amplitude $A$ of the output signal as the function of the parameter $k$ for different parameters $D=1$, 1.5, 2, and 3 with $b=1$, $E=1$, $a=1.5$, and $\omega=0.3$. The circle points are the results of numerical simulation in the case of $k=1.8$ with $b=1$, $E=1$, $a=1.5$, and $\omega=0.3$ for (a), and $D=1.5$ with $b=1$, $E=1$, $a=1.5$ and $\omega=0.3$ for (b).

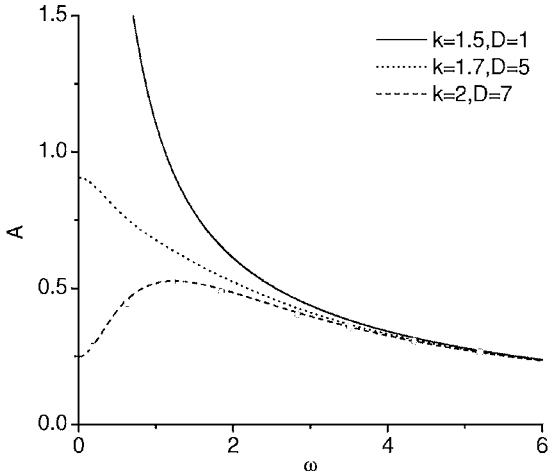


FIG. 3. The amplitude $A$ of the output signal as the function of the input signal frequency $\omega$ for the different parameter $k$ and noise strength $D$ ($k=1.5$ and $D=1$; $k=1.7$ and $D=5$; $k=2$ and $D=7$) when other parameters take $E=1$, $a=1.5$, and $b=1$. The circle points are the results of numerical simulation in the case of $k=2$ and $D=7$ with $E=1$, $a=1.5$, and $b=1$.

the parameter of the input signal, such as the input signal frequency $\omega$ (see the dashed curve in Fig. 3). In Fig. 1, we depict the dependence of the amplitude $A$ of the output signal on the noise parameters, namely, when the noise strength $D$ and the noise parameter $k$ vary simultaneously with other parameters $b=1$, $E=1$, $a=1.5$, and $\omega=0.5$. To observe the phenomena appearing in Fig. 1 clearly, in Fig. 2(a) we plot the objective figure of dependence of the amplitude $A$ of the output signal on the noise strength $D$ for different values $k$ ($k=1$, 1.8, 2.3, and 3) with the other parameters $b=1$, $E=1$, $a=1.5$, and $\omega=0.5$; in Fig. 2(b), we depict $A$ on the parameter $k$ describing the degree of the asymmetry of the dichotomous noise for different noise strength $D$ ($D=1$, 1.5, 2, and 3) when the other parameters take $b=1$, $E=1$, $a=1.5$, and $\omega=0.5$. In Fig. 3, we draw the figure of the amplitude $A$ as the function of the input signal frequency $\omega$ for the different values of noise strength $D$ and the parameter $k$ ($k=1.5$ and $D=1$; $k=1.7$ and $D=5$; and $k=2$ and $D=7$) when the other parameters take $E=1$, $a=1.5$, and $b=1$.

The results plotted in Figs. 1–3 are the results of long algebraic calculation. A comparison to the numerical simulation would help appreciate the correctness of the algebra. Thus, in Figs. 2(a), 2(b), and 3, we give some results of the numerical simulation (see the circle points in Figs. 2(a), 2(b), and 3) [19].

## III. CORRELATION FUNCTION AND SIGNAL-TO-NOISE RATIO

Generally speaking, we usually use the signal-to-noise ratio (SNR) to describe the SR phenomenon. Thus, it is necessary to calculate the SNR. To obtain that, we must first calculate the correlation function of the system

Multiplying $x(t')$ on both sides of Eq. (1), we can get

$$\frac{d[x(t)x(t')]}{dt} = [1 + a\sin(\omega t)]x(t') - (b-g)x(t)x(t') + \eta(t)x(t)x(t'). \quad (16)$$

Taking the average of Eq. (16) about $x$, we can obtain

$$\frac{d\langle x(t)x(t')\rangle}{dt} = [1 + a\sin(\omega t)]\langle x(t')\rangle - (b-g)\langle x(t)x(t')\rangle + \langle \eta(t)x(t)x(t')\rangle. \quad (17)$$

One new correlator $\langle \eta(t)x(t)x(t')\rangle$ appears in Eq. (17). To get the correlator, we will use the Shapiro-Loginov formula [16,18]. For this correlator $\langle \eta(t)x(t)x(t')\rangle$, we have

$$\frac{d\langle \eta(t)x(t)x(t')\rangle}{dt} = \left\langle \eta(t)\frac{d[x(t)x(t')]}{dt}\right\rangle - \lambda\langle \eta(t)x(t)x(t')\rangle + \lambda\langle \eta(t)\rangle\langle x(t)x(t')\rangle. \quad (18)$$

Substituting Eq. (16) into Eq. (18), we obtain

$$\frac{d\langle \eta(t)x(t)x(t')\rangle}{dt} = [1 + a\sin(\omega t)]\langle \eta(t)x(t')\rangle - (b-g+\lambda) \times\langle \eta(t)x(t)x(t')\rangle + (c^2 - \lambda g)\langle x(t)x(t')\rangle, \quad (19)$$

where we have used $\eta^2=c^2$. From Eqs. (17) and (19), we can obtain the two-order differential equation about $\langle x(t)x(t')\rangle$

$$\frac{d^2\langle x(t)x(t')\rangle}{dt^2} + [2(b-g)+\lambda]\frac{d\langle x(t)x(t')\rangle}{dt} + [(b+\lambda-g)(b-g) - c^2 + \lambda g]\langle x(t)x(t')\rangle = [a\omega\cos(\omega t) + (b+\lambda-g)(1 + a\sin(\omega t))]\langle x(t')\rangle + (1 + a\sin(\omega t))\langle \eta(t)x(t')\rangle. \quad (20)$$

According to the Shapiro-Loginov formula [16], the correlator $\langle \eta(t)x(t')\rangle$ in Eq. (20) satisfies

$$\frac{d\langle \eta(t)x(t')\rangle}{dt} = -\lambda\langle \eta(t)x(t')\rangle + \lambda\langle \eta(t)\rangle\langle x(t')\rangle. \quad (21)$$

Integrating Eq. (21) from $t'$ to $t$, and considering Eq. (9) in the calculation process, we can get the form of the solution of Eq. (21)

$$\begin{aligned}\langle \eta(t)x(t')\rangle &= \langle \eta(t')x(t')\rangle\exp(-\lambda|t-t'|) + g\langle x(t')\rangle \times[\exp(-\lambda|t-t'|) - 1] \\ &= \{A[\omega\cos(\varphi) + b\sin(\varphi)]\exp(-\lambda|t-t'|) - Ag\sin(\varphi)\}\cos(\omega t') \\ &\quad + \{[Ab\cos(\varphi) - A\omega\sin(\varphi) - a] \times\exp(-\lambda|t-t'|) - Ag\cos(\varphi)\}\sin(\omega t') \\ &\quad + (hb-1)\exp(-\lambda|t-t'|) - hg,\end{aligned} \quad (22)$$

where $h=\langle x(t)\rangle_{\text{inter}}$.

Substituting Eqs. (12) and (22) into Eq. (20), we get

$$\frac{d^2\langle x(t)x(t')\rangle}{dt^2}+[2(b-g)+\lambda]\frac{d\langle x(t)x(t')\rangle}{dt}+[(b+\lambda-g)(b-g)-c^2+\lambda g]\langle x(t)x(t')\rangle=a[\omega\cos(\omega t)+(b+\lambda-2g)\sin(\omega t)]\langle x(t')\rangle+(b+\lambda-2g)\langle x(t')\rangle+a[\langle x(t')\rangle'+b\langle x(t')\rangle-1-a\sin(\omega t')]\exp(-\lambda|t-t'|)\sin(\omega t)+[\langle x(t')\rangle'+b\langle x(t')\rangle-1-a\sin(\omega t')]\exp(-\lambda|t-t'|). \tag{23}$$

The solution of Eq. (23) is

$$\langle x(t)x(t')\rangle=M\sin(\omega t+\phi)+N\sin(\omega t+\psi)\exp(-\lambda|t-t'|)+D\exp(-\lambda|t-t'|)+F, \tag{24}$$

where

$$M=\left(\frac{w_3^2+w_4^2}{w_1^2+w_2^2}\right)^{1/2},\quad \phi=\arctan\left(\frac{w_1w_3-w_2w_4}{w_2w_3+w_1w_4}\right), \tag{25}$$

$$N=\left(\frac{z_3^2}{z_1^2+z_2^2}\right)^{1/2},\quad \psi=\arctan\left(\frac{z_2}{z_1}\right), \tag{26}$$

$$D=\frac{\langle x(t')\rangle'+b\langle x(t')\rangle-1-a\sin(\omega t')}{(b-\lambda-g)(b-g)-c^2+\lambda g}, \tag{27}$$

and

$$F=\frac{(b+\lambda-2g)\langle x(t')\rangle}{(b-g+\lambda)(b-g)-c^2+\lambda g} \tag{28}$$

with

$$w_1=(b-g+\lambda)(b-g)-c^2+\lambda g-\omega^2,$$

$$w_2=\omega(2(b-g)+\lambda),$$

$$w_3=a\omega\langle x(t')\rangle,\quad w_4=a(b+\lambda-2g)\langle x(t')\rangle, \tag{29}$$

and

$$z_1=(b-g-\lambda)(b-g)-c^2+\lambda g-\omega^2,\quad z_2=\omega[\lambda-2(b-g)],$$

$$z_3=a[\langle x(t')\rangle'+b\langle x(t')\rangle-1-a\sin(\omega t')]. \tag{30}$$

In Eq. (24), $|t-t'|=\tau>0$. Thus, we should calculate the correlation function when $t'=t+\tau$ and $t'=t-\tau$, respectively. Because the results of the calculation are more complicated, we write the results in Appendix B.

For Eq. (B1), the correlation function $\langle x(t)x(t-\tau)\rangle$ depends on both times $t$ and $\tau$. After taking the average about $t$ within one period of $2\pi\omega^{-1}$, we can obtain

$$\langle\langle x(t)x(t-\tau)\rangle\rangle_t=\frac{\omega}{2\pi}\int_0^{2\pi/\omega}\langle x(t)x(t-\tau)\rangle dt$$
$$=\frac{1}{2}\Bigg[\frac{aA[\omega w_1-(b+\lambda-2g)w_2]}{w_1^2+w_2^2}\sin(\varphi-\omega\tau)+\frac{aA[\omega w_2+(b+\lambda-2g)w_1]}{w_1^2+w_2^2}\cos(\varphi-\omega\tau)+\frac{a\exp(-\lambda\tau)}{z_1^2+z_2^2}\times(z_2\{A[\omega\cos(\varphi)+b\sin(\varphi)]\cos(\omega\tau)-[Ab\cos(\varphi)-A\omega\sin(\varphi)-a]\sin(\omega\tau)\}+z_1\{A[\omega\cos(\varphi)+b\sin(\varphi)]\sin(\omega\tau)+[Ab\cos(\varphi)-A\omega\sin(\varphi)-a]\cos(\omega\tau)\})\Bigg]+\frac{hb-1}{(b-\lambda-g)(b-g)-c^2+\lambda g}\exp(-\lambda\tau)+\frac{h(b+\lambda-2g)}{(b+\lambda-g)(b-g)-c^2+\lambda g}. \tag{31}$$

Similarly, after taking the average about $t$ within one period of $2\pi\omega^{-1}$ for $\langle x(t)x(t+\tau)\rangle$ (Eq. (B2)), we can get

$$\langle\langle x(t)x(t+\tau)\rangle\rangle_t=\frac{\omega}{2\pi}\int_0^{2\pi/\omega}\langle x(t)x(t+\tau)\rangle d(t+\tau)$$
$$=\frac{1}{2}\Bigg\{\frac{aA[\omega w_1-(b+\lambda-2g)w_2]}{w_1^2+w_2^2}\sin(\varphi-\omega\tau)+\frac{aA[\omega w_2+(b+\lambda-2g)w_1]}{w_1^2+w_2^2}\cos(\varphi-\omega\tau)+\frac{a\exp(-\lambda\tau)}{z_1^2+z_2^2}(z_2\{A[\omega\cos(\varphi)+b\sin(\varphi)]\cos(\omega\tau)-[Ab\cos(\varphi)-A\omega\sin(\varphi)-a]\sin(\omega\tau)\}+z_1\{A[\omega\cos(\varphi)+b\sin(\varphi)]\sin(\omega\tau)+[Ab\cos(\varphi)-A\omega\sin(\varphi)-a)\cos(\omega\tau)]\})+\frac{hb-1}{(b-\lambda-g)(b-g)-c^2+\lambda g}\exp(-\lambda\tau)+\frac{h(b+\lambda-2g)}{(b+\lambda-g)(b-g)-c^2+\lambda g}. \tag{32}$$

Comparing Eq. (31) to Eq. (32), we can find that the correlation functions $\langle\langle x(t)x(t-\tau)\rangle\rangle_t$ and $\langle\langle x(t)x(t+\tau)\rangle\rangle_t$ are the wholly same. This is the fact that the system is in its asymptotic limit. The power spectrum $S(\Omega)$ is defined as the Fourier transform of the correlation function. We first obtain one-side averaged power spectrum

$$\begin{aligned}S(\Omega)_{(+\Omega)} &= \int_{-\infty}^{0} \langle\langle x(t)x(t-\tau)\rangle\rangle_t \exp(-i\Omega\tau)d\tau + \int_{0}^{+\infty} \langle\langle x(t)x(t+\tau)\rangle\rangle_{t+\tau} \exp(-i\Omega\tau)d\tau \\ &= \frac{\pi}{2}\left(\frac{aA[\omega w_2+(b+\lambda-2g)w_1]}{w_1^2+w_2^2}\cos(\varphi)+\frac{aA[\omega w_1-(b+\lambda-2g)w_2]}{w_1^2+w_2^2}\sin(\varphi)\right)[\delta(\Omega+\omega)-\delta(\Omega-\omega)] \\ &\quad + \frac{a}{2(z_1^2+z_2^2)}(z_2A[\omega\cos(\varphi)+b\sin(\varphi)]+z_1[Ab\cos(\varphi)-A\omega\sin(\varphi)-a])\left[\frac{\lambda}{(\Omega-\omega)^2+\lambda^2}+\frac{\lambda}{(\Omega+\omega)^2+\lambda^2}\right] \\ &\quad + \frac{2\pi h(b+\lambda-2g)}{(b+\lambda-g)(b-g)-c^2+\lambda g}\delta(\Omega)+\frac{hb-1}{(b-\lambda-g)(b-g)-c^2+\lambda g}\frac{\lambda}{\Omega^2+\lambda^2}.\end{aligned} \tag{33}$$

By the same way, we can also obtain the other-side averaged power spectrum. Thus, the total power spectrum is

$$S(\Omega) = G_1\delta(\Omega-\omega)+G_2\delta(\Omega)+G_3, \tag{34}$$

where

$$G_1 = \pi\left\{\frac{aA[\omega w_2+(b+\lambda-2g)w_1]}{w_1^2+w_2^2}\cos(\varphi) + \frac{aA[\omega w_1-(b+\lambda-2g)w_2]}{w_1^2+w_2^2}\sin(\varphi)\right\},$$

$$G_2 = \frac{4\pi h(b+\lambda-2g)}{(b+\lambda-g)(b-g)-c^2+\lambda g},$$

$$\begin{aligned}G_3 &= \frac{a}{(z_1^2+z_2^2)}(z_2A[\omega\cos(\varphi)+b\sin(\varphi)] \\ &\quad + z_1[Ab\cos(\varphi)-A\omega\sin(\varphi)-a]) \\ &\quad \times\left[\frac{\lambda}{(\Omega-\omega)^2+\lambda^2}+\frac{\lambda}{(\Omega+\omega)^2+\lambda^2}\right] \\ &\quad + \frac{hb-1}{(b-\lambda-g)(b-g)-c^2+\lambda g}\frac{\lambda}{\Omega^2+\lambda^2}.\end{aligned} \tag{35}$$

We can note that the power spectrum in Eq. (34) is divided into three parts: the signal output, which is exhibited by a $\delta$ function at the frequency $\omega$; the zero-frequency term, which is connected with interaction between the asymmetric dichotomous noise and the internal dynamical of the system; and the broadband noise output term.

The SNR that we can obtain is

$$f_{\mathrm{SNR}} = \frac{G_1}{G_3(\Omega=\omega)}. \tag{36}$$

In Fig. 4, we plot the dependence of SNR on the noise strength $D$ and the noise parameter $k$, simultaneously, for the other parameters $E=1.0$, $\omega=0.3$, $a=1.5$, and $b=1.0$. It is clear that there is SR phenomenon. In addition, we also depict the dependence of SNR on the noise parameters $k$ and $D$, respectively [see Figs. 5(a) and 5(b)]. Some results of the numerical simulation [19] are also given in Figs. 5(a) and 5(b) [see the circle points in Figs. 5(a) and 5(b)].

## IV. CONCLUSION AND DISCUSSION

In this paper, we have derived the exact expression of the first moment, the correlation function, and the signal-to-noise ratio for a linear system subject to the multiplicative asymmetric dichotomous noise by making use of the "formula of differentiation" proposed by Shapiro and Loginov [16,18]. By studying the influence of the amplitude of the output signal of the first moment and the SNR as the functions of the parameters of the noise and the parameter of the input signal, we have found that the amplitude of the output signal and the SNR, respectively, have exhibited two kinds of SR phenomena. One is as the function of the parameters of the asymmetric dichotomous noise, such as the noise strength $D$ [see Figs. 1, 2(a), 4, and 5(a)], and the parameter $k$ [see Figs.

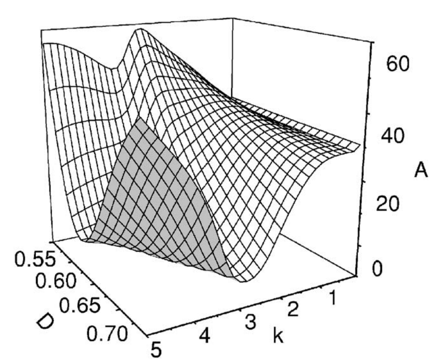


FIG. 4. The SNR as the function of the noise strength $D$ and the parameter $k$ for the other parameters $E=1$, $\omega=0.5$, $a=1.5$, and $b=1$.

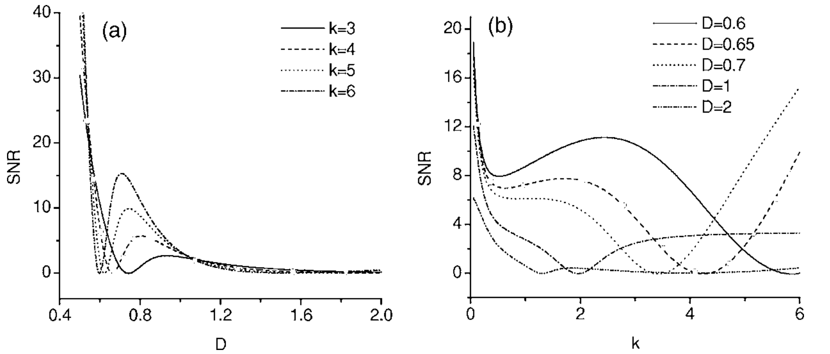


FIG. 5. (a) The SNR as the function of the noise strength $D$ for different parameters $k=3$, 4, 5, and 6 with $b=1$, $E=1$, $a=1.5$, and $\omega=0.5$; (b) the SNR the function of the parameter $k$ for different parameters $D=0.6$, 0.65, 0.7, 1 and 2 with $b=1$, $E=1$, $a=1.5$, and $\omega=0.5$. The circle points are the results of numerical simulation in the case of $k=4$ with $b=1$, $E=1$, $a=1.5$, and $\omega=0.5$ for (a), and $D=0.65$ with $b=1$, $E=1$, $a=1.5$, and $\omega=0.5$ for (b).

1, 2(b), 4, and 5(b)] describing the asymmetric degree of the dichotomous noise; the other is as the function of the parameter of the input signal, such as the input signal frequency $\omega$ (see the dashed curve in Fig. 3). We have noted that the SR phenomenon produced by the multiplicative noise has been first reported by Gammaitoni *et al.* [20]. (They call this phenomenon "Multiplicative stochastic resonance.") Moveover, SR in the presence of zero-mean asymmetric drives has also been studied in the literature [21]; in Ref. [22], Marchesoni *et al.* have investigated the effect of the asymmetry of an asymmetric Schmitt triger on the phenomenon of stochastic resonance and characterized this effect in terms of both spectral properties and switch time distributions of the trigger output.

In Figs. 2 and 5, we can observe that the phenomenon of stochastic resonance happens for different values of $D$ and $k$. This is because, for the fixed parameters $b$, $E$, $a$, and $\omega$, the intrinsic frequency of the system is determined by $D$ and $k$ [23]. In addition, in Fig. 5(b) we can observe the apparent divergence for large value of $k$. This is because, with increasing the value of $k$, for a large value of $k$, the broadband noise output for the power spectrum will become smaller and smaller (when $k\to\infty$, the noise output will tend to zero) [24].

The external periodic signal, a random force (noise), and nonlinearity are the necessary conditions for the appearance of SR. Nonlinearity means the system has two or more potential wells. SR can happen when the particles of the system are forced to move from one well to the other in the presence of the proper noise. However, there is only one potential well in a linear system. The additive noise can only change the internal thermal motion of particles and enhance the response of the system. It cannot affect the structure of the system. Thus a linear system, only driven by the additive noise, cannot give rise to SR phenomenon in the absence of the nonlinearity condition. Although multiplicative color noise belongs to the non-Markov noise, it can produce the nonlinearity which is needed for the appearance of the SR phenomenon. That is the reason why we find that SR takes place only for multiplicative color noise in a linear system.

The asymmetric dichotomous noise $\xi(t)$, which we assume it takes $a$ and $b$ ($a>b$) and $\langle\xi(t)\xi(t')\rangle=D\lambda\exp[-\lambda|t-t'|]$, can be transformed into the white (non-Gaussian) shot noise when $a\to\infty$, $\lambda\to\infty$, and $b$, $D$ fixed; it turns out to white Gaussian noise when $\lambda\to\infty$, $a\to\infty$, $b\to\infty$, and $D$ fixed. In studying these situations, we can find that no SR phenomena appear.

The system considered by us, in this paper, is driven by the multiplicative asymmetric dichotomous noise. It is found that SR phenomena appear. If the system is driven by additive asymmetric dichotomous noise, then no SR phenomenon appears. If the noise becomes three-states, or even more-states noise, then how is the case? This problem will be studied by us in the future.

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## APPENDIX A

In this appendix, we give an example of the application of Eq. (1) with noise $\xi(t)$ whose statistical properties satisfy Eqs. (2)–(4).

See the electric circuit in Fig. 6, where $L$ is a electric inductance (constant), $R$ a resistance (constant), and $r(t)$ a dichotomous resistor that fluctuates between two values $r_a$ and $r_b$, with mean waiting times $t_a$ and $t_b$, respectively. This can be achieved by inserting into the circuit a point contact whose conductance is controlled by an asymmetric two-level system tunneling incoherently between the two states with rates $\gamma=1/t_a$ and $\gamma'=1/t_b$ [25,26]. Accordingly, the fluctuations in the point contact resistor can be modeled as a stationary Markovian dichotomic process (telegraphic noise).

The dynamics of the circuit is governed by the following stochastic differential equation:

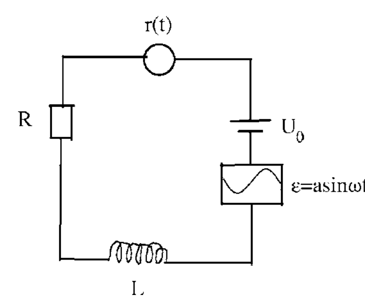


FIG. 6. An electric circuit considered by us. $R$ is a constant resistance; $L$ is a constant electric inductance; $r(t)$ is the dichotomous resistance, which has values $r_a$ and $r_b$ (the transition rate from $r_a$ to $r_b$ is $\gamma$, and the reverse transition rate is $\gamma'$); $U_0$ is a constant voltage source; and $\varepsilon=a\sin\omega t$ is an ac oscillatory voltage source.

$$L\frac{i}{dt} + Ri + ir(t) = U_0 + A\sin(\omega t), \tag{A1}$$

where $r(t)\in r_a, r_b$ ($r_a > r_b > 0$) is the dichotomous resistor, which is a telegraphic noise.

We assume $\gamma' = k\gamma$, i.e., $k = \gamma'/\gamma$. In order to get Eq. (1), we make the following transformation. Set $r(t) = \eta(t) + g$, where $\eta(t)$ is a dichotomous noise, which takes two values $E$ and $-kE$, $g$ is a constant, and the transition rate from $E$ to $-kE$ is $\gamma$ for $\eta(t)$, and the reverse transition rate is $\gamma'$. Using the relations between the noise $r(t)$ and the noise $\eta(t)$, we can get $E = (r_a - r_b)/(1+k)$ and $g = (kr_a + r_b)/(1+k)$. Taking the ensemble average of $r(t) = \eta(t) + g$, we can obtain $\langle \eta(t)\rangle = 0$. We can also derive the correlation function of $\eta(t)$: $\langle \eta(t)\eta(t')\rangle = D\lambda \exp[-\lambda|t-t'|]$.

Substituting $r(t) = \eta(t) + g$ into Eq. (A1), in the dimensionless form setting $L=1$, $U_0=1$ and $R+g=b$, and writing $A$, $i$ and $\eta(t)$ respectively as $a$, $x$ and $-\xi(t)$, we get

$$\dot{x} = 1 - bx + \xi(t)x + a\sin(\omega t), \tag{A2}$$

in which the statistical properties of the noise $\xi(t)$ satisfy Eqs. (2)–(4).

Equation (A2) is just the model we considered in this paper. Now model (1), studied in this paper, becomes a practical physical problem [i.e., an electric circuit (see Fig. 6)].

## APPENDIX B

When $t' = t - \tau$, Eq. (24) becomes

$$\begin{aligned}
\langle x(t)x(t-\tau)\rangle = {} & \frac{1}{2}\Bigg[\frac{aA[\omega w_1 - (b+\lambda-2g)w_2]}{w_1^2 + w_2^2}\sin(\varphi - \omega\tau) - \frac{aA[\omega w_2 + (b+\lambda-2g)w_1]}{w_1^2+w_2^2}\cos(\varphi-\omega\tau) \\
& + \frac{a\exp(-\lambda\tau)}{z_1^2+z_2^2}(z_2\{A[\omega\cos(\varphi) + b\sin(\varphi)]\cos(\omega\tau) - [Ab\cos(\varphi) - A\omega\sin(\varphi) - a]\sin(\omega\tau)\} - z_1\{A[\omega\cos(\varphi) \\
& + b\sin(\varphi)]\sin(\omega\tau) + [Ab\cos(\varphi) - A\omega\sin(\varphi) - a]\cos(\omega\tau)\})\Bigg]\cos(2\omega t) + \frac{1}{2}\Bigg[\frac{aA[\omega w_1 - (b+\lambda-2g)w_2]}{w_1^2+w_2^2} \\
& \times\cos(\varphi-\omega\tau) - \frac{aA[\omega w_2 + (b+\lambda-2g)w_1]}{w_1^2+w_2^2}\sin(\varphi-\omega\tau) + \frac{a\exp(-\lambda\tau)}{z_1^2+z_2^2}(z_1\{A[\omega\cos(\varphi) + b\sin(\varphi)]\cos(\omega\tau) \\
& - [Ab\cos(\varphi) - A\omega\sin(\varphi) - a]\sin(\omega\tau)\} + z_2\{A[\omega\cos(\varphi) + b\sin(\varphi)]\sin(\omega\tau) + [Ab\cos(\varphi) - A\omega\sin(\varphi) \\
& - a]\cos(\omega\tau)\})\Bigg]\sin(2\omega t) + \Bigg\{\frac{ah[\omega w_2 + (b+\lambda-2g)w_1]}{w_1^2+w_2^2} + \frac{a(hb-1)z_1\exp(-\lambda\tau)}{z_1^2+z_2^2} \\
& + \frac{A(b+\lambda-2g)\cos(\varphi-\omega\tau)}{(b+\lambda-g)(b-g)-c^2+\lambda g} + \frac{A[\omega\cos(\varphi)+b\sin(\varphi)]\sin(\omega\tau) + [Ab\cos(\varphi) - A\omega\sin(\varphi) - a]\cos(\omega\tau)}{(b-\lambda-g)(b-g)-c^2+\lambda g}\exp \\
& (-\lambda\tau)\Bigg\}\sin(\omega t) + \Bigg\{\frac{ah[\omega w_1 - (b+\lambda-2g)w_2]}{w_1^2+w_2^2} + \frac{a(hb-1)z_2\exp(-\lambda\tau)}{z_1^2+z_2^2} + \frac{A(b+\lambda-2g)\sin(\varphi-\omega\tau)}{(b+\lambda-g)(b-g)-c^2+\lambda g} \\
& + \frac{A[\omega\cos(\varphi)+b\sin(\varphi)]\cos(\omega\tau) - [Ab\cos(\varphi) - A\omega\sin(\varphi) - a]\sin(\omega\tau)}{(b-\lambda-g)(b-g)-c^2+\lambda g}\exp(-\lambda\tau)\Bigg\}\cos(\omega t) \\
& + \frac{1}{2}\Bigg[\frac{aA[\omega w_1 - (b+\lambda-2g)w_2]}{w_1^2+w_2^2}\sin(\varphi-\omega\tau) + \frac{aA[\omega w_2 + (b+\lambda-2g)w_1]}{w_1^2+w_2^2}\cos(\varphi-\omega\tau) \\
& + \frac{a\exp(-\lambda\tau)}{z_1^2+z_2^2}(z_2\{A[\omega\cos(\varphi)+b\sin(\varphi)]\cos(\omega\tau) - [Ab\cos(\varphi) - A\omega\sin(\varphi) - a]\sin(\omega\tau)\} + z_1\{A[\omega\cos(\varphi) \\
& + b\sin(\varphi)]\sin(\omega\tau) + [Ab\cos(\varphi) - A\omega\sin(\varphi) - a]\cos(\omega\tau)\})\Bigg] + \frac{hb-1}{(b-\lambda-g)(b-g)-c^2+\lambda g}\exp(-\lambda\tau) \\
& + \frac{h(b+\lambda-2g)}{(b+\lambda-g)(b-g)-c^2+\lambda g}.
\end{aligned} \tag{B1}$$

However, when $t' = t + \tau$, the Eq. (24) becomes

$$
\begin{aligned}
\langle x(t)x(t+\tau)\rangle = {}& \frac{1}{2}\bigg[\frac{aA[\omega w_1-(b+\lambda-2g)w_2]}{w_1^2+w_2^2}\sin(\varphi-\omega\tau)-\frac{aA[\omega w_2+(b+\lambda-2g)w_1]}{w_1^2+w_2^2}\cos(\varphi-\omega\tau)\\
&-\frac{a\exp(-\lambda\tau)}{z_1^2+z_2^2}(z_2\{A[\omega\cos(\varphi)+b\sin(\varphi)]\cos(\omega\tau)-[Ab\cos(\varphi)-A\omega\sin(\varphi)-a]\sin(\omega\tau)\}\\
&-z_1\{A[\omega\cos(\varphi)+b\sin(\varphi)]\sin(\omega\tau)+[Ab\cos(\varphi)-A\omega\sin(\varphi)-a)\cos(\omega\tau\,]\})\bigg]\cos[2\omega(t+\tau)]\\
&+\frac{1}{2}\bigg[\frac{aA[\omega w_1-(b+\lambda-2g)w_2]}{w_1^2+w_2^2}\cos(\varphi-\omega\tau)+\frac{aA[\omega w_2+(b+\lambda-2g)w_1]}{w_1^2+w_2^2}\sin(\varphi-\omega\tau)\\
&+\frac{a\exp(-\lambda\tau)}{z_1^2+z_2^2}(z_1\{A[\omega\cos(\varphi)+b\sin(\varphi)]\cos(\omega\tau)-[Ab\cos(\varphi)-A\omega\sin(\varphi)-a]\sin(\omega\tau)\}+z_2\{A[\omega\cos(\varphi)\\
&+b\sin(\varphi)]\sin(\omega\tau)+[Ab\cos(\varphi)-A\omega\sin(\varphi)-a]\cos(\omega\tau)\})\bigg]\sin[2\omega(t+\tau)]\\
&+\bigg\{\frac{a(hb-1)z_1\exp(-\lambda\tau)}{z_1^2+z_2^2}\cos(\omega\tau)+\frac{a(hb-1)z_2\exp(-\lambda\tau)}{z_1^2+z_2^2}\sin(\omega\tau)+\frac{ah[\omega w_2+(b+\lambda-2g)w_1]}{w_1^2+w_2^2}\cos(\omega\tau)\\
&+\frac{ah[\omega w_1-(b+\lambda-2g)w_2]}{w_1^2+w_2^2}\sin(\omega\tau)+\frac{A(b+\lambda-2g)\cos(\varphi)}{(b+\lambda-g)(b-g)-c^2+\lambda g}\\
&+\frac{(Ab\cos(\varphi)-A\omega\sin(\varphi)-a)\exp(-\lambda\tau)}{(b-\lambda-g)(b-g)-c^2+\lambda g}\bigg\}\sin(\omega(t+\tau))+\bigg\{-\frac{ah[\omega w_2+(b+\lambda-2g)w_1]}{w_1^2+w_2^2}\sin(\omega\tau)\\
&+\frac{ah[\omega w_1-(b+\lambda-2g)w_2]}{w_1^2+w_2^2}\cos(\omega\tau)-\frac{a(hb-1)z_1\exp(-\lambda\tau)}{z_1^2+z_2^2}\sin(\omega\tau)+\frac{a(hb-1)z_2\exp(-\lambda\tau)}{z_1^2+z_2^2}\cos(\omega\tau)\\
&+\frac{A(b+\lambda-2g)\sin(\varphi)}{(b+\lambda-g)(b-g)-c^2+\lambda g}+\frac{[A\omega\cos(\varphi)+Ab\sin(\varphi)]\exp(-\lambda\tau)}{(b-\lambda-g)(b-g)-c^2+\lambda g}\bigg\}\cos[\omega(t+\tau)]\\
&+\frac{1}{2}\bigg[\frac{aA[\omega w_1-(b+\lambda-2g)w_2]}{w_1^2+w_2^2}\sin(\varphi-\omega\tau)+\frac{aA[\omega w_2+(b+\lambda-2g)w_1]}{w_1^2+w_2^2}\cos(\varphi-\omega\tau)+\frac{a\exp(-\lambda\tau)}{z_1^2+z_2^2}\\
&\times(z_2\{A[\omega\cos(\varphi)+b\sin(\varphi)]\cos(\omega\tau)-[Ab\cos(\varphi)-A\omega\sin(\varphi)-a)\sin(\omega\tau)]\;+z_1\{A[\omega\cos(\varphi)\\
&+b\sin(\varphi)]\sin(\omega\tau)+[Ab\cos(\varphi)-A\omega\sin(\varphi)-a)\cos(\omega\tau)]\})\bigg]+\frac{hb-1}{(b-\lambda-g)(b-g)-c^2+\lambda g}\exp(-\lambda\tau)\\
&+\frac{h(b+\lambda-2g)}{(b+\lambda-g)(b-g)-c^2+\lambda g}.
\end{aligned}
\tag{B2}
$$


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[24] The system has two states. One is when $\xi(t)$ takes the value $-E$, the other is when $\xi(t)$ takes $kE$. The probability density for $\xi(t)=-E$ is $P(-E)=\frac{k}{1+k}$; the probability density for $\xi(t)=kE$ is $P(kE)=\frac{1}{1+k}$. It is clear that, in the limit of $k\to\infty$, $P(-E)\to 1$ and $P(kE)\to 0$. Thus, in the limit of $k\to\infty$, the noise $\xi(t)$ will tend to a constant $-E$, and the noise output will tend to zero. Moreover, we know that SNR is defined as $\mathrm{SNR}=\frac{f_s}{f_N}$, in which $f_s$ is the signal output and $f_N$ the noise output for the power spectrum of the system. Thus, for a large value of $k$, SNR will have the apparent divergence (in the limit of $k\to\infty$, $f_N\to 0$).

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